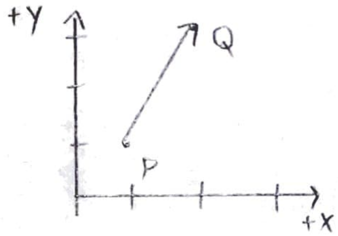
Lecture 2: Vectors

A vector is a "directed line segment" in space. It is often convenient to imagine them as arrows pointing in a certain direction.

Ex. 1 Draw the vector pointing from $P=(1,1)$ to $Q=(2,3)$



It is often more useful to represent vectors algebraically instead of geometrically. To do so, we recognize the important features of a vector are its direction and magnitude. These can be represented by the vectors "components".

$$P=(x_0, y_0, z_0) \quad \& \quad Q=(x_1, y_1, z_1)$$

$$\overrightarrow{PQ} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$$

Ex. 2 Given $P=(3,-2,1)$ & $Q=(-2,5,3)$ find $\overrightarrow{PQ}$

$$\overrightarrow{PQ} = \langle (-2-3), (5-2), (3-1) \rangle = \langle -5, 7, 2 \rangle$$

Vector Combination

To add or subtract vectors, we do so component-wise. Given:

$$\bar{u} = \langle u_1, u_2, u_3 \rangle \qquad \bar{v} = \langle v_1, v_2, v_3 \rangle$$

$$\bar{u} + \bar{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$$

$$\bar{u} - \bar{v} = \langle u_1 - v_1, u_2 - v_2, u_3 - v_3 \rangle$$

Vector Scaling

A "number" as we normally think of them are merely 1D vectors and we call them "scalars". Given a scalar c,

$$c\bar{u} = \langle cu_1, cu_2, cu_3 \rangle$$

Ex. 3 Given $\bar{u} = \langle 1, -1, 3 \rangle$ & $\bar{v} = \langle 2, 2, -1 \rangle$, find $\bar{u} + \bar{v}$ and $\bar{u} - \bar{v}$.

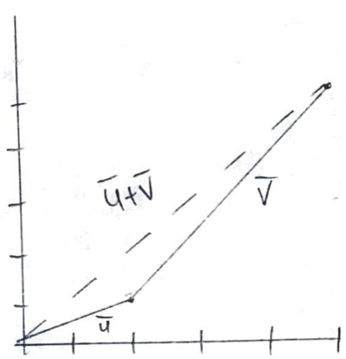
$\bar{u} + \bar{v} = \langle 1+2, -1+2, 3+{}^{-}1 \rangle = \langle 3, 1, 2 \rangle$

$\bar{u} - \bar{v} = \langle 1-2, -1-2, 3-{}^{-}1 \rangle = \langle -1, -3, 4 \rangle$

Geometric Interpretation

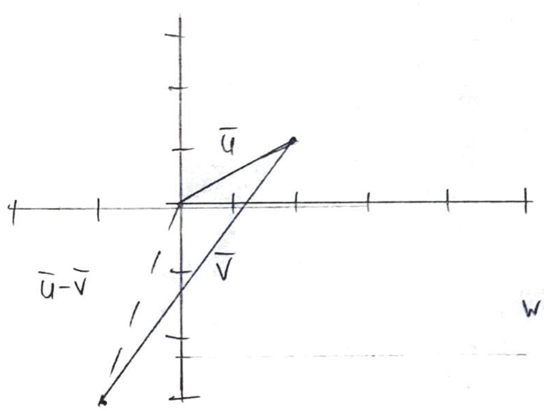
It is sometimes convenient to show vector addition/subtraction graphically. Mostly in physics or engineering applications.

Ex. 4 Given $\bar{u} = \langle 2, 1 \rangle$ & $\bar{v} = \langle 3, 4 \rangle$ find $\bar{u}+\bar{v}$ & $\bar{u}-\bar{v}$ graphically.



$\bar{u} + \bar{v} = \langle 5, 5 \rangle$

The addition is performed by attaching the tail of vector 2 to the head of vector 1.



$\bar{u} - \bar{v} = \langle -1, -3 \rangle$

The subtraction is done similarly. Except we attached $-\bar{v}$ to the head of $\bar{u}$

Ex. 5 Given $\bar{u} = \langle 2, 4 \rangle$, $v = \langle 1, 3 \rangle$, and $c = 5$ find $c\bar{u} - \bar{v}$ and $\bar{u} + c\bar{v}$

$c\bar{u} - \bar{v} = 5\langle 2, 4 \rangle - \langle 1, 3 \rangle = \langle 10, 8 \rangle - \langle 1, 3 \rangle = \langle 9, 5 \rangle$

$\bar{U} + c\bar{V} = \langle 2, 4 \rangle + 5\langle 1, 3 \rangle = \langle 2, 4 \rangle + \langle 5, 15 \rangle = \langle 7, 19 \rangle$

Vector Norm

The length or "norm" of a vector, $\bar{u}$, is denoted by $\|u\|$. Given,

$\bar{u} = \langle u_1, u_2, u_3 \rangle$

$$\|\bar{u}\| = \sqrt{u_1^2 + u_2^2 + u_3^2}$$

Ex. 6 Find $\|\bar{u}\|$ & $\|\bar{v}\|$ when $\bar{u} = \langle 4, 0, 3 \rangle$ & $\bar{v} = \langle -2, 1, 5 \rangle$

$$\|\bar{u}\| = \sqrt{4^2 + 0^2 + 3^2} = \sqrt{25} = 5$$

$$\|\bar{v}\| = \sqrt{(-2)^2 + 1^2 + 5^2} = \sqrt{30}$$

Ex. 7 Find $\|u\|$ when $\bar{u} = \langle 3, 2 \rangle$

$$\|\bar{u}\| = \sqrt{3^2 + 2^2} = \sqrt{13}$$

Unit Vector

A vector with length 1, or "unity", is called a unit vector. Any vector can be turned into a unit vector.

Ex. 8 Find the unit vector pointing in the same direction as $\bar{u} = \langle 3, 5 \rangle$.

$$\hat{u} = \frac{\bar{u}}{\|u\|} = \frac{1}{\sqrt{3^2+5^2}} \langle 3, 5 \rangle = \left\langle \frac{3}{\sqrt{34}}, \frac{5}{\sqrt{34}} \right\rangle$$

The "hat" in $\hat{u}$ denotes a unit vector.

$\hat{i}$ $\hat{j}$ $\hat{k}$

There are three special unit vectors: $\hat{i}$, $\hat{j}$, & $\hat{k}$.

$$\hat{i} = \langle 1, 0, 0 \rangle$$

$$\hat{j} = \langle 0, 1, 0 \rangle$$

$$\hat{k} = \langle 0, 0, 1 \rangle$$

These vectors allow us to represent vectors as linear combinations of $\hat{i}$, $\hat{j}$, & $\hat{k}$

Ex. 9 Write $\bar{u} = \langle 5, 7, 2 \rangle$ in $\hat{i}$, $\hat{j}$, $\hat{k}$ notation

$$\bar{u} = \langle 5, 7, 2 \rangle = 5\hat{i} + 7\hat{j} + 2\hat{k}$$

Zero Vector

The "zero vector" is a vector with no length or direction. It is denoted as:

$$\overline{\emptyset} = \langle \emptyset, \emptyset, \emptyset \rangle$$

Parallel Vectors

Two vectors, $\bar{u}$ & $\bar{v}$, are parallel if and only if there is a scalar, c, such that:

$$\bar{u} = c\bar{v}$$

Ex. 1Ø Are $\bar{u} = \langle 3, 5, 7 \rangle$ and $\bar{v} = \langle 9, 15, 21 \rangle$ parallel?

Yes! $\bar{u} = \frac{1}{3}\bar{v}$